

The maximal p -norm multiplicativity conjecture is false

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For all $1 < p < 2$, we demonstrate the existence of quantum channels with non-multiplicative maximal p -norms. Equivalently, the minimum output Renyi entropy of order p of a quantum channel is not additive for all $1 < p < 2$. The violations found are large. As p approaches 1, the minimum output Renyi entropy of order p for a product channel need not be significantly greater than the minimum output entropy of its individual factors. Since $p = 1$ corresponds to the von Neumann entropy, these counterexamples demonstrate that if the additivity conjecture of quantum information theory is true, it cannot be proved as a consequence of maximal p -norm multiplicativity.

I. INTRODUCTION

The oldest problem of quantum information theory is arguably to determine the capacity of a quantum-mechanical communications channel for carrying information, specifically “classical” *bits* of information. (Until the 1990’s it would have been unnecessary to add that additional qualification, but today the field is equally concerned with other forms of information like *qubits* and *ebits* that are fundamentally quantum-mechanical.) The classical capacity problem long predates the invention of quantum source coding [1, 2] and was of concern to the founders of information theory themselves [3]. The first major result on the problem came with the resolution of a conjecture of Gordon’s [4] by Alexander Holevo in 1973, when he published the first proof [5] that the maximum amount of information that can be extracted from an ensemble of states ρ_i occurring with probabilities p_i is bounded above by

$$\chi(\{p_i, \rho_i\}) = H\left(\sum_i p_i \rho_i\right) - \sum_i p_i H(\rho_i), \quad (1)$$

where $H(\rho) = -\text{Tr } \rho \ln \rho$ is the von Neumann entropy of the density operator ρ . For a quantum channel \mathcal{N} , one can then define the Holevo capacity

$$\chi(\mathcal{N}) = \max_{\{p_i, \rho_i\}} \chi(\{p_i, \mathcal{N}(\rho_i)\}), \quad (2)$$

where the maximization is over all ensembles of input states. Writing $C(\mathcal{N})$ for the classical capacity of the channel \mathcal{N} , this leads easily to an upper bound of

$$C(\mathcal{N}) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \chi(\mathcal{N}^{\otimes n}). \quad (3)$$

It then took more than two decades for further substantial progress to be made on the problem, but in 1996, building on recent advances [6], Holevo [7] and Schumacher-Westmoreland [8] managed to show that the upper bound in Eq. (3) is actually achieved. This was a resolution of sorts to the capacity problem, but the limit in the equation makes it in practice extremely difficult to evaluate. If the codewords used for data transmission are restricted such that they are not entangled across

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multiple uses of the channel, however, the resulting *product state capacity* $C_{1\infty}(\mathcal{N})$ has the simpler expression

$$C_{1\infty}(\mathcal{N}) = \chi(\mathcal{N}). \quad (4)$$

The additivity conjecture for the Holevo capacity asserts that for all channels \mathcal{N}_1 and \mathcal{N}_2 ,

$$\chi(\mathcal{N}_1 \otimes \mathcal{N}_2) = \chi(\mathcal{N}_1) + \chi(\mathcal{N}_2). \quad (5)$$

This would imply, in particular, that $C_{1\infty}(\mathcal{N}) = C(\mathcal{N})$, or that entangled codewords do not increase the classical capacity of a quantum channel.

In 2003, Peter Shor [9], building on several previously established connections [10, 11, 12], demonstrated that the additivity of the Holevo capacity, the additivity of the entanglement of formation [13, 14, 15, 16] and the superadditivity of the entanglement of formation [17] are all equivalent to yet another conjecture, known as the *minimum entropy output conjecture* [18], which is particularly simple to express mathematically. For a channel \mathcal{N} , define

$$H^{\min}(\mathcal{N}) = \min_{|\varphi\rangle} H(\mathcal{N}(\varphi)), \quad (6)$$

where the minimization is over all pure input states $|\varphi\rangle$. The minimum entropy output conjecture asserts that for all channels \mathcal{N}_1 and \mathcal{N}_2 ,

$$H^{\min}(\mathcal{N}_1 \otimes \mathcal{N}_2) = H^{\min}(\mathcal{N}_1) + H^{\min}(\mathcal{N}_2). \quad (7)$$

There has been a great deal of previous work on these conjectures, particularly numerical searches for counterexamples, necessarily in low dimension, at Caltech, IBM, IMAPh and by ER-ATO researchers [19], as well as proofs of many special cases. For example, the minimum entropy output conjecture has been shown to hold if one of the channels is the identity channel [20, 21], a unital qubit channel [22], a generalized depolarizing channel [23, 24] or an entanglement-breaking channel [9, 25, 26]. In addition, the weak additivity conjecture was confirmed for degradable channels [27], their conjugate channels [28] and some other special classes of channels [16, 29, 30, 31]. Further evidence for qubit channels was supplied in [18]. This list is by no means exhaustive. The reader is directed to Holevo's reviews for a detailed account of the history of the additivity problem [32, 33].

For the past several years, the most commonly used strategy for proving these partial results has been to demonstrate the multiplicativity of maximal p -norms of quantum channels for p approaching 1 [20]. For a quantum channel \mathcal{N} and $p > 1$, define the maximal p -norm of \mathcal{N} to be

$$\nu_p(\mathcal{N}) = \sup \left\{ \|\mathcal{N}(\rho)\|_p ; \rho \geq 0, \text{Tr } \rho = 1 \right\}. \quad (8)$$

In the equation, $\|\sigma\|_p = (\text{Tr } |\sigma|^p)^{1/p}$. The *maximal p -norm multiplicativity conjecture* [20] asserts that for all quantum channels \mathcal{N}_1 and \mathcal{N}_2 ,

$$\nu_p(\mathcal{N}_1 \otimes \mathcal{N}_2) = \nu_p(\mathcal{N}_1) \nu_p(\mathcal{N}_2). \quad (9)$$

This can be re-expressed in an equivalent form more convenient to us using Renyi entropies. Define the Renyi entropy of order p to be

$$H_p(\rho) = \frac{1}{1-p} \ln \text{Tr } \rho^p \quad (10)$$

for $p > 0$, $p \neq 1$. Since $\lim_{p \downarrow 1} H_p(\rho) = H(\rho)$, we will also define $H_1(\rho)$ to be $H(\rho)$. All these entropies have the property that they are 0 for pure states and achieve their maximum value of the logarithm of the dimension on maximally mixed states. Define the minimum output Renyi entropy H_p^{\min} by substituting H_p for H in Eq. (6). Eq. (9) can then be written equivalently as

$$H_p^{\min}(\mathcal{N}_1 \otimes \mathcal{N}_2) = H_p^{\min}(\mathcal{N}_1) + H_p^{\min}(\mathcal{N}_2), \quad (11)$$

which underscores the fact that the maximal p -norm multiplicativity conjecture is a natural strengthening of the original minimum entropy output conjecture (7).

This conjecture spawned a significant literature of its own which we will not attempt to summarize. Holevo's reviews are again an excellent source [32, 33]. Some more recent important references include [34, 35, 36, 37, 38]. Unlike the von Neumann entropy case, however, some counterexamples had already been found prior to this paper. Namely, Werner and Holevo found a counterexample to Eq. (11) for $p > 4.79$ [39] that nonetheless doesn't violate the p -norm multiplicativity conjecture for $1 < p < 2$ [40], and very recently Winter showed that the conjecture is false for all $p > 2$ [41]. In light of these developments, the standing conjecture was that the maximal p -norm multiplicativity held for $1 \leq p \leq 2$, corresponding to the region in which the map $X \mapsto X^p$ is operator convex [34]. More conservatively, it was conjectured to hold at least in an open interval $(1, 1 + \epsilon)$, which would be sufficient to imply the minimum entropy output conjecture.

On the contrary, we will show that the conjecture is false for all $1 < p < 2$. In particular, given $1 < p < 2$, we show that there exist channels \mathcal{N}_1 and \mathcal{N}_2 with output dimension d such that both $H_p^{\min}(\mathcal{N}_1)$ and $H_p^{\min}(\mathcal{N}_2)$ are equal to $\ln d - \mathcal{O}(1)$ but $H_p^{\min}(\mathcal{N}_1 \otimes \mathcal{N}_2) = p \ln d + \mathcal{O}(1)$. Thus,

$$H_p^{\min}(\mathcal{N}_1) + H_p^{\min}(\mathcal{N}_2) - H_p^{\min}(\mathcal{N}_1 \otimes \mathcal{N}_2) = (2 - p) \ln d - \mathcal{O}(1). \quad (12)$$

For p close to 1, one finds that the minimum entropy output of the product channel need not be significantly larger than the minimum output entropy of the individual factors. Since [20, 23]

$$H_p^{\min}(\mathcal{N}_1 \otimes \mathcal{N}_2) \geq H_p^{\min}(\mathcal{N}_1) = \ln d - \mathcal{O}(1), \quad (13)$$

these counterexamples are essentially the strongest possible for p close to 1.

At $p = 1$ itself, however, we see no evidence of a violation of the additivity conjecture for the channels we study. Thus, the conjecture stands and it is still an open question whether entangled codewords can increase the classical capacity of a quantum channel.

Notation: If A and B are finite dimensional Hilbert spaces, we write $AB \equiv A \otimes B$ for their tensor product and $|A|$ for $\dim A$. The Hilbert spaces on which linear operators act will be denoted by a superscript. For instance, we write φ^{AB} for a density operator on AB . Partial traces will be abbreviated by omitting superscripts, such as $\varphi^A \equiv \text{Tr}_B \varphi^{AB}$. We use a similar notation for pure states, e.g. $|\psi\rangle^{AB} \in AB$, while abbreviating $\psi^{AB} \equiv |\psi\rangle\langle\psi|^{AB}$. We associate to any two isomorphic Hilbert spaces $A \simeq A'$ a *unique* maximally entangled state which we denote $|\Phi\rangle^{AA'}$. Given any orthonormal basis $\{|i\rangle^A\}$ for A , if we define $|i\rangle^{A'} = V|i\rangle^A$ where V is the associated isomorphism, we can write this state as $|\Phi\rangle^{AA'} = |A|^{-1/2} \sum_{i=1}^{|A|} |i\rangle^A |i\rangle^{A'}$. We will also make use of the asymptotic notation $f(n) = \mathcal{O}(g(n))$ if there exists $C > 0$ such that for sufficiently large n , $|f(n)| \leq Cg(n)$. $f(n) = \Omega(g(n))$ is defined similarly but with the reverse inequality $|f(n)| \geq Cg(n)$. Finally, $f(n) = \Theta(g(n))$ if $f(n) = \mathcal{O}(g(n))$ and $f(n) = \Omega(g(n))$.

II. THE COUNTEREXAMPLES

Let E, F and G be finite dimensional quantum systems, then define $R = E, S = FG, A = EF$ and $B = G$, so that $RS = AB = EFG$. Our counterexamples will be channels from S to A of the form

$$\mathcal{N}(\rho) = \text{Tr}_B [U(|0\rangle\langle 0|^R \otimes \rho)U^\dagger] \quad (14)$$

for U unitary and $|0\rangle$ some fixed state on R . Our method will be to fix the dimensions of the systems involved, select U at random, and show that the resulting channel is likely to violate additivity. The rough intuition motivating our examples will be to exploit the fact that there are channels that appear to be highly depolarizing for product state inputs despite the fact that they are not close to the depolarizing channel in, for example, the norm of complete boundedness [42].

Consider a single copy of \mathcal{N} and the associated map $|0\rangle^R |\varphi\rangle^S \mapsto U|0\rangle^R |\varphi\rangle^S$. This map takes S to a subspace of $A \otimes B$, and if U is selected according to the Haar measure, then the image of S is itself a random subspace, distributed according to the unitarily invariant measure. In [43], it was shown that if $|S|$ is chosen appropriately, then the image is likely to contain only almost maximally entangled states, as measured by the entropy of entanglement. After tracing over B , this entropy of entanglement becomes the entropy of the output state. Thus, for S of suitable size, all input states get mapped to high entropy output states. We will repeat the analysis below, finding that the maximum allowable size of S will depend on p as described by the following lemma:

Lemma II.1 *Let A and B be quantum systems with $2 \leq |A| \leq |B|$ and $1 < p < 2$. Then there exists a subspace $S \subset A \otimes B$ of dimension*

$$|S| = \left\lfloor \frac{\Gamma_p |A|^{2-p} |B| \alpha^{2.5}}{p-1} \right\rfloor, \quad (15)$$

with $\Gamma_p > 0$ a constant, that contains only states $|\varphi\rangle \in S$ with high entanglement, in the sense that

$$H_p(\varphi^A) \geq \ln |A| - \alpha - \beta, \quad (16)$$

where $\beta = 2|A|/|B|$. The probability that a subspace of dimension $|S|$ chosen at random according to the unitarily invariant measure will not have this property is bounded above by

$$\left(\frac{|A|^{(p-1)/2}}{\alpha} \right)^{2|S|} \exp \left(- \frac{(2|A||B| - 1)\alpha^2}{2|A|^{p-1}} \right). \quad (17)$$

Proof The argument is nearly identical to the proof of Theorem IV.1 in [43] so we will only discuss the differences here, referring the reader to the original paper to complete the argument. The first ingredient is the estimate $\mathbb{E}H_p(\varphi^A) \geq \mathbb{E}H_2(\varphi^A) \geq \ln |A| - |A|/|B|$, where the expectation is over random pure states on $A \otimes B$ [44]. All that is required in addition is an upper bound on the Lipschitz constant of the maps $f_p(|\varphi\rangle) = H_p(\varphi^A)$ for $p > 1$. Let $|\varphi\rangle = \sum_{jk} \varphi_{jk} |jk\rangle^{AB}$. We begin, as in [43], by bounding the Lipschitz constant of $g_p(|\varphi\rangle) = H_p(\sum_j \langle j|\varphi^A|j\rangle)$. For $q_j = \sum_k |\varphi_{jk}|^2$,

$$\nabla g_p \cdot \nabla g_p = \frac{4p^2}{(1-p)^2} \frac{\sum_j q_j^{2p-1}}{(\sum_j q_j^p)^2} \leq \frac{4p^2}{(1-p)^2} \frac{1}{\sum_j q_j^p} \leq \frac{4p^2}{(1-p)^2} |A|^{p-1}, \quad (18)$$

using the facts that $q_j^{2p-1} \leq q_j^p$ for $p > 1$ in the first inequality and that $\sum_j q_j^p$ is minimized by the uniform distribution in the second. Eq. (18) therefore provides an upper bound on the square

of the Lipschitz constant of g_p . The Schur concavity of the Renyi entropies then ensures that the same argument as was used in [43] can be used to upper bound the Lipschitz constant of f_p by that of g_p . \square

Now consider the product channel $\mathcal{N} \otimes \bar{\mathcal{N}}$, where $\bar{\mathcal{N}}(\rho) = \text{Tr}_B [\bar{U}(|0\rangle\langle 0| \otimes \rho)U^T]$. We will exploit a form of the same symmetry as Werner-Holevo [39] and Winter [41] did, but instead of using an exact symmetry that occurs only rarely, we'll use an approximate version of it that holds always. In the trivial case where $|R| = 1$, the identity $U \otimes \bar{U}|\Phi\rangle = (U\bar{U}^T \otimes I)|\Phi\rangle = |\Phi\rangle$ for the maximally entangled state $|\Phi\rangle^{S_1 S_2}$ implies that

$$(\mathcal{N} \otimes \bar{\mathcal{N}})(|\Phi\rangle\langle\Phi|^{S_1 S_2}) = \text{Tr}_{B_1 B_2} [|\Phi\rangle\langle\Phi|^{A_1 A_2} \otimes |\Phi\rangle\langle\Phi|^{B_1 B_2}] = |\Phi\rangle\langle\Phi|^{A_1 A_2}. \quad (19)$$

The output of $\mathcal{N} \otimes \bar{\mathcal{N}}$ will thus be a pure state. In the general case, we will choose R to be small but not trivial, in which case useful bounds can still be placed on the largest eigenvalue of the output state for an input state maximally entangled between S_1 and S_2 .

Lemma II.2 *Let $|\Phi\rangle^{S_1 S_2}$ be a state maximally entangled between S_1 and S_2 as in the previous paragraph. Then $(\mathcal{N} \otimes \bar{\mathcal{N}})(\Phi^{S_1 S_2})$ has an eigenvalue of at least $\frac{|S|}{|A||B|}$.*

Proof This is an easy calculation again exploiting the $U \otimes \bar{U}$ invariance of the maximally entangled state. Recall that $R = E$, $S = FG$, $A = EF$ and $B = G$:

$$A_1 A_2 \langle\Phi|(\mathcal{N} \otimes \bar{\mathcal{N}})(\Phi^{S_1 S_2})|\Phi\rangle^{A_1 A_2} \quad (20)$$

$$= \text{Tr} \left[(\Phi^{A_1 A_2} \otimes I^{B_1 B_2})(U \otimes \bar{U})(|00\rangle\langle 00|^{R_1 R_2} \otimes \Phi^{S_1 S_2})(U^\dagger \otimes U^T) \right] \quad (21)$$

$$\geq \text{Tr} \left[(U^\dagger \otimes U^T)(\Phi^{E_1 E_2} \otimes \Phi^{F_1 F_2} \otimes \Phi^{G_1 G_2})(U \otimes \bar{U})(|00\rangle\langle 00|^{E_1 E_2} \otimes \Phi^{F_1 F_2} \otimes \Phi^{G_1 G_2}) \right] \quad (22)$$

$$= \text{Tr} \left[(\Phi^{E_1 E_2} \otimes \Phi^{F_1 F_2} \otimes \Phi^{G_1 G_2})(|00\rangle\langle 00|^{E_1 E_2} \otimes \Phi^{F_1 F_2} \otimes \Phi^{G_1 G_2}) \right] \quad (23)$$

$$= \frac{1}{|E|} = \frac{|S|}{|A||B|}. \quad (24)$$

Note that U acts on $E_1 F_1 G_1$ and \bar{U} on $E_2 F_2 G_2$. In the third line we have used the operator inequality $\Phi^{G_1 G_2} \leq I^{G_1 G_2}$ and the cyclic property of the trace. \square

In order to demonstrate violations of additivity, the first step is to bound the minimum output entropy from below for a single copy of the channel. Fix $1 < p < 2$, let $|B| = 2|A|$ so that $\beta = 1$, set $\alpha = 1$, and then choose $|S|$ according to Lemma II.1. With probability approaching 1 as $|A| \rightarrow \infty$, when U is chosen according to the Haar measure,

$$H_p^{\min}(\mathcal{N}) \geq \ln |A| - 2. \quad (25)$$

The same obviously holds for $H_p^{\min}(\bar{\mathcal{N}})$. Recall that the entropy of the uniform distribution is $\ln |A|$ so the minimum entropy is nearly maximal.

On the other hand, by Lemma II.2,

$$H_p((\mathcal{N} \otimes \bar{\mathcal{N}})(\Phi)) \leq \frac{1}{1-p} \ln \left(\frac{|S|}{|A||B|} \right)^p = \frac{p}{1-p} \ln \frac{|S|}{|A||B|}. \quad (26)$$

Substituting the same value of $|S|$ into this inequality yields

$$H_p((\mathcal{N} \otimes \bar{\mathcal{N}})(\Phi)) \leq p \ln |A| + \mathcal{O}(1). \quad (27)$$

Since $p < 2$, the Renyi entropy of $(\mathcal{N} \otimes \bar{\mathcal{N}})(\Phi)$ is strictly less than $H_p^{\min}(\mathcal{N}) + H_p^{\min}(\bar{\mathcal{N}}) \geq 2 \ln |A| - \mathcal{O}(1)$, where the last inequality holds with high probability. This is a violation of conjecture (11), with the size of the gap approaching $\ln |A| - \mathcal{O}(1)$ as p tends to 1.

Theorem II.3 *For all $1 < p < 2$, there exists a quantum channel for which the inequalities (25) and (27) both hold. The inequalities are inconsistent with the maximal p -norm multiplicativity conjecture.*

Note, however, that changing p also requires changing $|S|$ according to Lemma II.1, so we have a sequence of channels violating additivity of the minimal output Renyi entropy as p decreases to 1, as opposed to a single channel doing so for every p . This prevents us from drawing conclusions about the von Neumann entropy by taking the limit $p \rightarrow 1$.

As an aside, it is interesting to observe that violating maximal p -norm multiplicativity has structural consequences for the channels themselves. For example, because entanglement-breaking channels do not violate multiplicativity [45], there must be states $|\psi\rangle^{S_1 S_2}$ such that $(\mathcal{N} \otimes I^{S_2})(\psi)$ is entangled, despite the fact that \mathcal{N} will be a rather noisy channel. (The same conclusions apply to the channels used as examples by Winter [41], where the conclusion takes the form that ϵ -randomizing maps need not be entanglement-breaking.)

III. THE VON NEUMANN ENTROPY CASE

Despite the large violations found for p close to 1, the class of examples presented here do not appear to contradict the minimum entropy output conjecture for the von Neumann entropy. The reason is that the upper bound demonstrated for $H_p((\mathcal{N} \otimes \bar{\mathcal{N}})(\Phi))$ in the previous section rested entirely on the existence of one large eigenvalue for $(\mathcal{N} \otimes \bar{\mathcal{N}})(\Phi)$. The von Neumann entropy is not as sensitive to the value of a single eigenvalue as are the Renyi entropies for $p > 1$ and, consequently, does not appear to exhibit additivity violations. With a bit of work, it is possible to make these observations more rigorous.

Lemma III.1 *Let $|\Phi\rangle^{S_1 S_2}$ be a maximally entangled state between S_1 and S_2 . Assuming that $|A| \leq |B| \leq |S|$,*

$$\int \text{Tr} \left[((\mathcal{N} \otimes \bar{\mathcal{N}})(|\Phi\rangle\langle\Phi|))^2 \right] dU = \frac{|S|^2}{|A|^2 |B|^2} + \mathcal{O}\left(\frac{1}{|A|^2}\right), \quad (28)$$

where “ dU ” is the normalized Haar measure on $R \otimes S \cong A \otimes B$.

A description of the calculation can be found in Appendix A. Let the eigenvalues of $(\mathcal{N} \otimes \bar{\mathcal{N}})(\Phi)$ be equal to $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{|A|^2}$. For a typical U , Lemmas II.2 and III.1 together imply that

$$\sum_{j>1} \lambda_j^2 = \mathcal{O}\left(\frac{1}{|A|^2}\right). \quad (29)$$

Thus, aside from λ_1 , the eigenvalues λ_j must be quite small. A typical eigenvalue distribution is plotted in Figure 1. If we define $\tilde{\lambda}_j = \lambda_j / (1 - \lambda_1)$, then $\sum_{j>1} \tilde{\lambda}_j = 1$ and

$$H_1(\tilde{\lambda}) \geq H_2(\tilde{\lambda}) = -\ln \sum_{j>1} \tilde{\lambda}_j^2 = 2 \ln |A| - \mathcal{O}(1). \quad (30)$$

An application of the grouping property then gives us a good lower bound on the von Neumann entropy:

$$H_1((\mathcal{N} \otimes \bar{\mathcal{N}})(\Phi)) = H_1(\lambda) = h(\lambda_1) + (1 - \lambda_1)H_1(\tilde{\lambda}) = 2 \ln |A| - \mathcal{O}(1), \quad (31)$$

where h is the binary entropy function. This entropy is nearly as large as it can be and, in particular, as large as $H^{\min}(\mathcal{N}) + H^{\min}(\bar{\mathcal{N}})$ according to Theorem IV.1 of [43], the von Neumann entropy version of Lemma II.1.

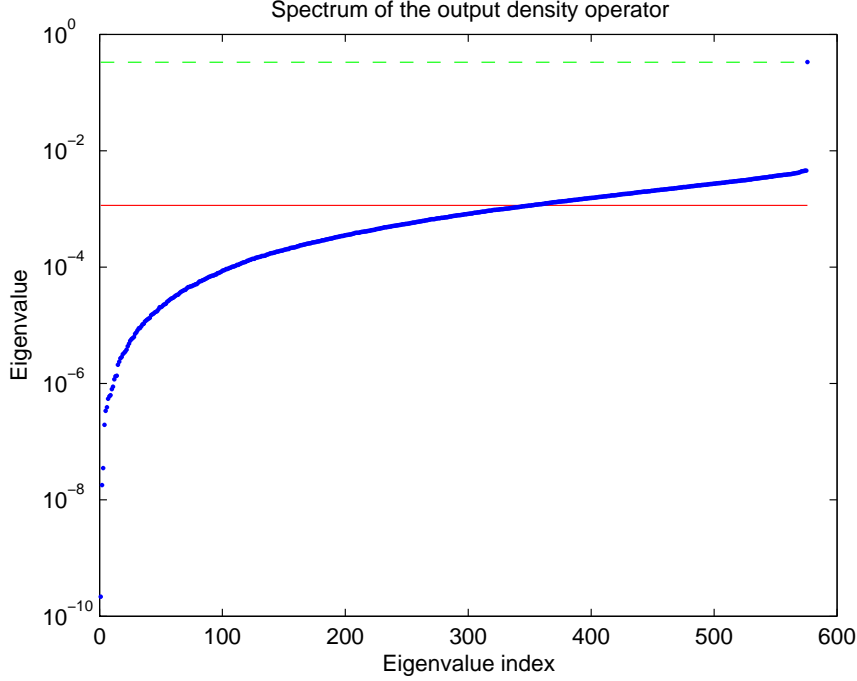


FIG. 1: Typical eigenvalue spectrum of $(\mathcal{N} \otimes \bar{\mathcal{N}})(\Phi)$ when $|R| = 3$ and $|A| = |B| = 24$. The eigenvalues are plotted in increasing order from left to right. The green dashed line corresponds to $|S|/(|A||B|) = 1/3$, which is essentially equal to the largest eigenvalue. The red solid line represents the value $(1 - \frac{|S|}{|A||B|})/|A|^2 = 1/864$. If the density operator were maximally mixed aside from its largest eigenvalue, all but that one eigenvalue would fall on this line. While that is not the case here or in general, the remaining eigenvalues are nonetheless sufficiently small to ensure that the density operator has high von Neumann entropy.

IV. DISCUSSION

The counterexamples presented here demonstrate that the maximal p -norm multiplicativity conjecture is false for $1 < p < 2$. The primary motivation for studying this conjecture was that it is a natural strengthening of the minimum entropy output conjecture, which is of fundamental importance in quantum information theory. In particular, since the multiplicativity conjecture was formulated, most attempts to prove the minimum entropy output conjecture for special cases actually proved maximal p -norm multiplicativity and then took the limit as p decreases to 1. This strategy, we now know, cannot be used to prove the conjecture in general.

From that perspective, it would seem that the results in this paper cast doubt on the validity of the minimum entropy output conjecture itself. However, as we have shown, the examples explored here appear to be completely consistent with the conjecture, precisely because the von Neumann entropy is more difficult to perturb than the Renyi entropies of order $p > 1$. Indeed, the message of this paper may be that attempts to prove the minimum entropy output conjecture have all been approaching $p = 1$ from the wrong direction. It is quite possible that additivity of the minimum output Renyi entropy holds for $0 \leq p \leq 1$ and then fails dramatically for $p > 1$.

This is not, unfortunately, a very well-informed speculation. With few exceptions [46], there has been very little research on the additivity question in the regime $p < 1$, even though many arguments can be easily adapted to this parameter region. (Eq. (13), for example, holds for all $0 < p$.) Remedying this oversight would now seem to be a priority.

Note: In the short time since I circulated a preliminary version of this manuscript, there has already been substantial further progress. Most notably, Andreas Winter has observed that the Lipschitz bound in Lemma II.1 is not tight. By improving it, he managed to disprove the p -norm multiplicativity conjecture at its final redoubt of $p = 2$ and simultaneously demonstrate that the large violation observed here near $p = 1$ can be found for all $p > 1$. Also, Frédéric Dupuis confirmed that replacing the unitary group by the orthogonal group leads to qualitatively similar conclusions. Those and other developments will appear in [47].

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APPENDIX A: PROOF OF LEMMA III.1

We will estimate the integral, in what is perhaps not the most illuminating way, by expressing it in terms of the matrix entries of U . Let $U_{s,ab} = {}^R\langle 0|^S \langle s|U|a\rangle^A |b\rangle^B$. Expanding gives

$$\begin{aligned} & \int \text{Tr} \left[((\mathcal{N} \otimes \bar{\mathcal{N}})(|\Phi\rangle\langle\Phi|))^2 \right] dU \\ &= \frac{1}{|S|^2} \sum_{\substack{a_1, a_2 \\ a'_1, a'_2}} \sum_{\substack{b_1, b_2 \\ b'_1, b'_2}} \sum_{\substack{s_1, s_2 \\ s'_1, s'_2}} \int \bar{U}_{s_1, a_2 b_2} \bar{U}_{s_2, a'_1 b'_1} \bar{U}_{s'_1, a'_2 b'_2} \bar{U}_{s'_2, a_1 b_1} U_{s_1, a_1 b_1} U_{s_2, a'_2 b_2} U_{s'_1, a'_1 b'_1} U_{s'_2, a_2 b'_2} dU. \end{aligned} \quad (\text{A1})$$

Following [48, 49], the non-zero terms in the sum can be represented using a simple graphical notation. Make two parallel columns of four dots, then label the left-hand dots by the indices (s_1, s_2, s'_1, s'_2) and the right-hand dots by the indices $\vec{v} = (a_2 b_2, a'_1 b_1, a'_2 b'_2, a_1 b'_1)$. Join dots with a solid line if the corresponding \bar{U} matrix entry appears in Eq. (A1). Since terms integrate to a non-zero value only if the vector of U indices $\vec{w} = (a_1 b_1, a'_2 b_2, a'_1 b'_1, a_2 b'_2)$ is a permutation of the vector of \bar{U} indices, a non-zero integral can be represented by using a dotted line to connect left-hand and right-hand dots whenever the corresponding U matrix entry appears in the integral.

Assuming for the moment that the vertex labels in the left column are all distinct and likewise for the right column, the integral evaluates to the Weingarten function $\text{Wg}(\pi)$, where π is the permutation such that $w_i = v_{\pi(i)}$. For the rough estimate required here, it is sufficient to know that $\text{Wg}(\pi) = \Theta((|A||B|)^{-4-|\pi|})$, where $|\pi|$ is the minimal number of factors required to write π as a product of transpositions, and that $\text{Wg}(e) = (|A||B|)^{-4}(1 + \mathcal{O}(|A|^{-2}|B|^{-2}))$ [50].

The dominant contribution to Eq. (A1) comes from the “stack” diagram

$$\begin{aligned}
s_1 &\bullet \text{---} \bullet a_2 b_2 = a_1 b_1 \\
s_2 &\bullet \text{---} \bullet a'_1 b_1 = a'_2 b_2 \\
s'_1 &\bullet \text{---} \bullet a'_2 b'_2 = a'_1 b'_1 \\
s'_2 &\bullet \text{---} \bullet a_1 b'_1 = a_2 b'_2,
\end{aligned}$$

in which the solid and dashed lines are parallel and for which the contribution is positive and approximately equal to

$$\frac{1}{|S|^2} \sum_{\substack{a_1, a_2 \\ a'_1, a'_2}} \sum_{\substack{b_1, b_2 \\ b'_1, b'_2}} \sum_{\substack{s_1, s_2 \\ s'_1, s'_2}} \delta_{a_1 a_2} \delta_{b_1 b_2} \delta_{a'_1 a'_2} \delta_{b'_1 b'_2} \text{Wg}(\text{id}) = \frac{|S|^2}{|A|^2 |B|^2} (1 + \mathcal{O}(|A|^{-2} |B|^{-2})). \quad (\text{A2})$$

(The expression on the left-hand side would be exact but for the terms in which vertex labels are not distinct.) To obtain an estimate of Eq. (A1), it is then sufficient to examine the other terms and confirm that they are all of smaller asymptotic order than this. There are six diagrams representing transpositions, and their associated (negative) contributions are

$s_1 \bullet \text{---} \bullet a_2 b_2 = a_1 b_1$ $s_2 \bullet \text{---} \bullet a'_1 b_1 = a'_2 b_2$ $s'_1 \bullet \text{---} \bullet a'_2 b'_2 = a_2 b'_2$ $s'_2 \bullet \text{---} \bullet a_1 b'_1 = a'_1 b'_1$ $\Theta(S ^2 A ^{-4} B ^{-2})$	$s_1 \bullet \text{---} \bullet a_2 b_2 = a_1 b_1$ $s_2 \bullet \text{---} \bullet a'_1 b_1 = a_2 b'_2$ $s'_1 \bullet \text{---} \times \bullet a'_2 b'_2 = a'_1 b'_1$ $s'_2 \bullet \text{---} \bullet a_1 b'_1 = a'_2 b_2$ $\Theta(S ^2 A ^{-4} B ^{-4})$	$s_1 \bullet \text{---} \bullet a_2 b_2 = a_2 b'_2$ $s_2 \bullet \text{---} \bullet a'_1 b_1 = a'_2 b_2$ $s'_1 \bullet \text{---} \times \bullet a'_2 b'_2 = a'_1 b'_1$ $s'_2 \bullet \text{---} \bullet a_1 b'_1 = a_1 b_1$ $\Theta(S ^2 A ^{-2} B ^{-4})$
$s_1 \bullet \text{---} \bullet a_2 b_2 = a_1 b_1$ $s_2 \bullet \text{---} \bullet a'_1 b_1 = a'_1 b'_1$ $s'_1 \bullet \text{---} \bullet a'_2 b'_2 = a'_2 b_2$ $s'_2 \bullet \text{---} \bullet a_1 b'_1 = a_2 b'_2$ $\Theta(S ^2 A ^{-2} B ^{-4})$	$s_1 \bullet \text{---} \bullet a_2 b_2 = a'_1 b'_1$ $s_2 \bullet \text{---} \times \bullet a'_1 b_1 = a'_2 b_2$ $s'_1 \bullet \text{---} \bullet a'_2 b'_2 = a_1 b_1$ $s'_2 \bullet \text{---} \bullet a_1 b'_1 = a_2 b'_2$ $\Theta(S ^2 A ^{-4} B ^{-4})$	$s_1 \bullet \text{---} \bullet a_2 b_2 = a'_2 b_2$ $s_2 \bullet \text{---} \times \bullet a'_1 b_1 = a_1 b_1$ $s'_1 \bullet \text{---} \bullet a'_2 b'_2 = a'_1 b'_1$ $s'_2 \bullet \text{---} \bullet a_1 b'_1 = a_2 b'_2$ $\Theta(S ^2 A ^{-4} B ^{-2})$

For permutations π such that $|\pi| > 1$, the Weingarten function is significantly suppressed: $\text{Wg}(\pi) = \mathcal{O}(|A|^{-6} |B|^{-6})$. Moreover, for a given diagram type, the requirement that $w_i = v_{\pi(i)}$ can only hold if at least two pairs of the indices $a_1, a_2, a'_1, a'_2, b_1, b_2, b'_1, b'_2$ are identical. The contribution from such diagrams is therefore $\mathcal{O}(|S|^2 |A|^{-4} |B|^{-2})$.

To finish the proof, it is necessary to consider integrals in which the vertex labels on the left- or the right-hand side of a diagram are not all distinct. In this more general case, choosing a set \mathcal{C} of representatives for the conjugacy classes of the permutation group on four elements, the value of the integral can be written

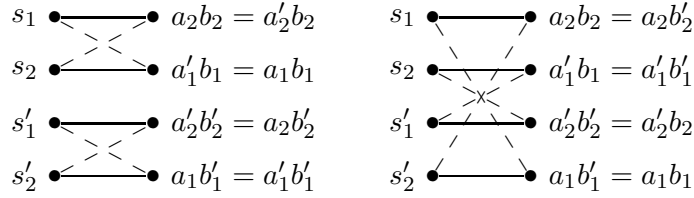
$$\sum_{c \in \mathcal{C}} N(c) \text{Wg}(c), \quad (\text{A3})$$

where

$$N(c) = \sum_{\substack{\sigma \in \mathcal{S}_4: \\ \vec{v} = \sigma(\vec{v})}} \sum_{\substack{\tau \in \mathcal{S}_4: \\ \vec{w} = \tau(\vec{w})}} \delta(\tau \pi \sigma \in c). \quad (\text{A4})$$

These formulas have a simple interpretation. Symmetry in the vertex labels introduces ambiguities in the diagrammatic notation; the formula states that every one of the diagrams consistent with a given vertex label set must be counted, and with a defined dimension-independent multiplicity. Conveniently, our crude estimates have already done exactly that, ignoring the multiplicities. The only case for which we need to know the multiplicities, moreover, is for contributions to the dominant term, which we want to know exactly and not just up to a constant multiple.

We claim that in the sum (A1) there are at most $\mathcal{O}(|S|^4|A||B|^3)$ terms with vertex label symmetry. The total contribution for terms with vertex label symmetries τ and σ in which $|\tau\pi\sigma| \geq 1$ is therefore of size $\mathcal{O}(|S|^2|A|^{-4}|B|^{-2})$ and does not affect the dominant term. To see why the claim holds, fix a diagram type and recall that the requirement $w_i = v_{\pi(i)}$ for a permutation π can only hold if at least two pairs of the indices $a_1, a_2, a'_1, a'_2, b_1, b_2, b'_1, b'_2$ are identical. Equality is achieved only when all the A indices or all the B indices are aligned, corresponding to the following two diagrams:



For the first diagram, using the fact that $|A| \leq |B| \leq |S|$, it is easy to check that imposing the extra constraint that either the top or bottom two S or AB vertex labels match singles at most $\mathcal{O}(|S|^4|A||B|^3)$ terms from Eq. (A1). Similar reasoning applies to the second diagram, but imposing the constraint instead on rows one and four, or two and three. For all other diagram types, at least four pairs of the indices $a_1, a_2, a'_1, a'_2, b_1, b_2, b'_1, b'_2$ are identical. (The number of matching A and B indices is necessarily even.) In a term for which the vertex labels are not all distinct, either a pair of S indices or a further pair of A or B indices must be identical. In the latter case, there must exist an identical A pair *and* an identical B pair among all the pairs. Again using $|A| \leq |B| \leq |S|$, there can be at most $\mathcal{O}(|S|^4|B|^3)$ such terms per diagram type, which demonstrates the claim.

We are thus left to consider integrals with vertex label symmetry and $N(e) \neq 0$ in Eq. (A3). If $N(e) = 1$, then our counting was correct and there is no problem. It is therefore sufficient to bound the number of integrals in which $N(e) > 1$. This can occur only in terms with at least 2 vertex label symmetries. Running the argument of the previous paragraph again, for the two diagrams with A or B indices all aligned, this occurs in at most $\mathcal{O}(|S|^4|B|^2)$ terms. For the rest of the cases, it is necessary to impose equality on yet another pair of indices, leading again to at most $\mathcal{O}(|S|^4|B|^2)$ terms. Since $\text{Wg}(e) = \mathcal{O}(|A|^{-4}|B|^{-4})$, these contributions are collectively $\mathcal{O}(|S|^2|A|^{-4}|B|^{-2})$.

The bound on the error term in Eq. (28) arises by substituting the inequalities $|S| \leq |A||B|$ and $|A| \leq |B|$ into each of the estimates calculated above.

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- [1] B. Schumacher. Quantum coding. *Physical Review A*, 51:2738–2747, 1995.
 - [2] R. Jozsa and B. Schumacher. A new proof of the quantum noiseless coding theorem. *J. Mod. Opt.*, 41:2343–2349, 1994.
 - [3] J. Pierce. The early days of information theory. *IEEE Transactions on Information Theory*, 19(1):3–8, 1973.
 - [4] J. P. Gordon. Noise at optical frequencies; information theory. In P. A. Miles, editor, *Quantum electronics and coherent light; Proceedings of the international school of physics Enrico Fermi, Course XXXI*, pages 156–181, New York, 1964. Academic Press.

- [5] A. S. Holevo. Information theoretical aspects of quantum measurements. *Probl. Info. Transm. (USSR)*, 9(2):31–42, 1973. Translation: *Probl. Info. Transm.* vol. 9, pp. 177–183, 1973.
- [6] P. Hausladen, R. Jozsa, B. Schumacher, M. Westmoreland, and W. K. Wootters. Classical information capacity of a quantum channel. *Physical Review A*, 54:1869–1876, 1996.
- [7] A. S. Holevo. The capacity of the quantum channel with general signal states. *IEEE Trans. Inf. Theory*, 44:269–273, 1998.
- [8] B. Schumacher and M. D. Westmoreland. Sending classical information via noisy quantum channels. *Physical Review A*, 56:131–138, 1997.
- [9] P. W. Shor. Additivity of the classical capacity of entanglement-breaking quantum channels. *Journal of Mathematical Physics*, 43:4334–4340, 2002. arXiv:quant-ph/0201149.
- [10] K. M. R. Audenaert and S. L. Braunstein. On strong superadditivity of the entanglement of formation. *Communications in Mathematical Physics*, 246:443–452, 2004. arXiv:quant-ph/0303045.
- [11] K. Matsumoto, T. Shimono, and A. Winter. Remarks on additivity of the Holevo channel capacity and of the entanglement of formation. *Communications in Mathematical Physics*, 246:427–442, 2004. arXiv:quant-ph/0206148.
- [12] A. A. Pomeransky. Strong superadditivity of the entanglement of formation follows from its additivity. *Physical Review A*, 68(3):032317–+, September 2003. arXiv:quant-ph/0305056.
- [13] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters. Mixed-state entanglement and quantum error correction. *Physical Review A*, 54:3824–3851, 1996. arXiv:quant-ph/9604024.
- [14] P. M. Hayden, M. Horodecki, and B. M. Terhal. The asymptotic entanglement cost of preparing a quantum state. *Journal of Physics A: Mathematical and General*, 34:6891–6898, 2001. arXiv:quant-ph/0008134.
- [15] G. Vidal, W. Dür, and J. I. Cirac. Entanglement cost of bipartite mixed states. *Physical Review Letters*, 89(2):027901–+, 2002. arXiv:quant-ph/0112131.
- [16] K. Matsumoto and F. Yura. Entanglement cost of antisymmetric states and additivity of capacity of some quantum channels. *Journal of Physics A: Mathematical and General*, 37:L167–L171, 2004. arXiv:quant-ph/0306009.
- [17] K. G. H. Vollbrecht and R. F. Werner. Entanglement measures under symmetry. *Physical Review A*, 64(6):062307–+, 2001. arXiv:quant-ph/0010095.
- [18] C. King and M. B. Ruskai. Minimal entropy of states emerging from noisy quantum channels. *IEEE Transactions on Information Theory*, 47(1):192–209, 2001. arXiv:quant-ph/9911079.
- [19] S. Osawa and H. Nagaoka. Numerical experiments on the capacity of quantum channel with entangled input states. *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, E84-A(10):2583–2590, 2001. arXiv:quant-ph/0007115.
- [20] G. G. Amosov, A. S. Holevo, and R. F. Werner. On some additivity problems of quantum information theory. *Probl. Inform. Transm.*, 36(4):25, 2000.
- [21] G. G. Amosov and A. S. Holevo. On the multiplicativity conjecture for quantum channels. March 2001. arXiv:math-ph/0103015.
- [22] C. King. Additivity for unital qubit channels. *Journal of Mathematical Physics*, 43(10):4641–4643, 2002. arXiv:quant-ph/0103156v1.
- [23] C. King. The capacity of the quantum depolarizing channel. *IEEE Transactions on Information Theory*, 49(1):221–229, 2003. arXiv:quant-ph/0204172.
- [24] A. Fujiwara and T. Hashizumé. Additivity of the capacity of depolarizing channels. *Physics Letters A*, 299:469–475, July 2002.
- [25] A. S. Holevo. Quantum coding theorems. *Russ. Math. Surv.*, 53:1295–1331, 1998.
- [26] C. King. Maximization of capacity and p-norms for some product channels. arXiv:quant-ph/0103086, 2001.
- [27] I. Devetak and P. W. Shor. The capacity of a quantum channel for simultaneous transmission of classical and quantum information. *Communications in Mathematical Physics*, 256:287–303, June 2005.
- [28] C. King, K. Matsumoto, M. Nathanson, and M. B. Ruskai. Properties of conjugate channels with applications to additivity and multiplicativity. arXiv:quant-ph/0509126, 2005.
- [29] N. Datta, A. S. Holevo, and Y. M. Suhov. A quantum channel with additive minimum output entropy. 2004. arXiv:quant-ph/0403072.
- [30] J. Cortese. Holevo-Schumacher-Westmoreland channel capacity for a class of qudit unital channels. *Physical Review A*, 69(2):022302–+, 2004.
- [31] M. Fukuda. Extending additivity from symmetric to asymmetric channels. *Journal of Physics A: Math-*

- ematical and General*, 38:L753–L758, November 2005. arXiv:quant-ph/0505022.
- [32] A. S. Holevo. Additivity of classical capacity and related problems. Available online at: <http://www.imaph.tu-bs.de/qi/problems/10.pdf>, 2004.
 - [33] A. S. Holevo. The additivity problem in quantum information theory. In *Proceedings of the International Congress of Mathematicians, Madrid, Spain, 2006*, Publ. EMS, pages 999–1018, Zurich, 2007.
 - [34] C. King and M. B. Ruskai. Comments on multiplicativity of maximal p -norms when $p = 2$. *Quantum Information and Computation*, 4:500–512, 2004. arXiv:quant-ph/0401026.
 - [35] C. King and M. B. Ruskai. Multiplicativity properties of entrywise positive maps. *Linear algebra and its applications*, 404:367–379, 2005. arXiv:quant-ph/0409181.
 - [36] A. Serafini, J. Eisert, and M. M. Wolf. Multiplicativity of maximal output purities of gaussian channels under gaussian inputs. *Phys. Rev. A*, 71(1):012320–+, January 2005.
 - [37] I. Devetak, M. Junge, C. King, and M. B. Ruskai. Multiplicativity of completely bounded p -norms implies a new additivity result. *Communications in Mathematical Physics*, 266:37–63, August 2006.
 - [38] S. Michalakis. Multiplicativity of the maximal output 2-norm for depolarized werner-holevo channels. arXiv:0707.1722, 2007.
 - [39] R. F. Werner and A. S. Holevo. Counterexample to an additivity conjecture for output purity of quantum channels. *Journal of Mathematical Physics*, 43:4353–4357, 2002. arXiv:quant-ph/0203003.
 - [40] N. Datta. Multiplicativity of maximal p -norms in Werner-Holevo channels for $1 < p < 2$. arXiv:quant-ph/0410063, 2004.
 - [41] A. Winter. The maximum output p -norm of quantum channels is not multiplicative for any $p > 2$. arXiv:0707.0402, 2007.
 - [42] V. I. Paulsen. *Completely bounded maps and dilations*. Longman scientific and technical, New York, 1986.
 - [43] P. Hayden, D. W. Leung, and A. Winter. Aspects of generic entanglement. *Communications in Mathematical Physics*, 265:95–117, 2006. arXiv:quant-ph/0407049.
 - [44] E. Lubkin. Entropy of an n -system from its correlation with a k -reservoir. *J. Math. Phys.*, 19:1028, 1978.
 - [45] C. King. Maximal p -norms of entanglement breaking channels. *Quantum information and computation*, 3(2):186–190, 2003. arXiv:quant-ph/0212057.
 - [46] M. M. Wolf and J. Eisert. Classical information capacity of a class of quantum channels. *New Journal of Physics*, 7:93–+, 2005. arXiv:quant-ph/0412133.
 - [47] F. Dupuis, P. Hayden, D. Leung, and A. Winter. The maximal p -norm multiplicativity conjecture is really, really false. In preparation, 2007.
 - [48] S. Aubert and C. S. Lam. Invariant integration over the unitary group. *Journal of Mathematical Physics*, 44:6112–6131, 2003. arXiv:math-ph/0307012.
 - [49] S. Aubert and C. S. Lam. Invariant and group theoretical integrations over the $U(n)$ group. *Journal of Mathematical Physics*, 45:3019–3039, 2004. arXiv:math-ph/0405036.
 - [50] B. Collins and P. Śniady. Integration with respect to the Haar measure on unitary, orthogonal and symplectic group. *Communications in Mathematical Physics*, 264:773–795, 2006. arXiv:math-ph/0402073.